On Stability of Heegaard Splittings
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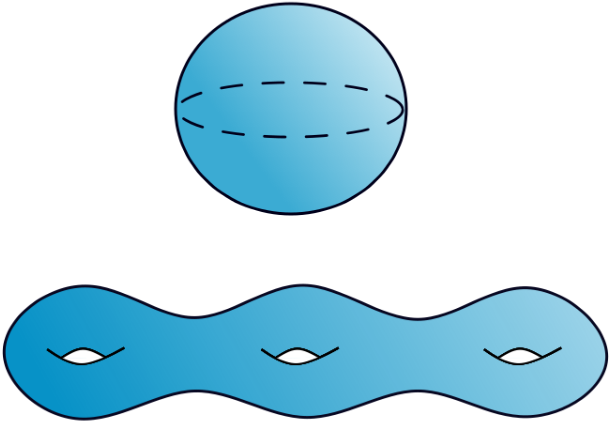
Outline

- Background and definitions
  - 2-Manifolds and 3-Manifolds
  - Compression Body
    - Handlebody
    - Trivial Compression Body
  - Heegaard Splitting
  - Stabilization
- Stabilization Theorem
  - Proof Idea

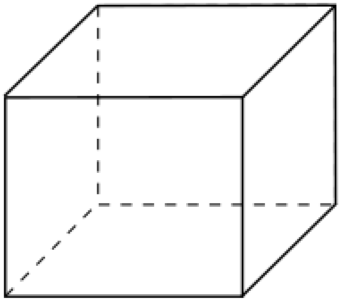
2-Manifolds

Def: Topological spaces which are locally homeomorphic to $\mathbb{R}^2$.

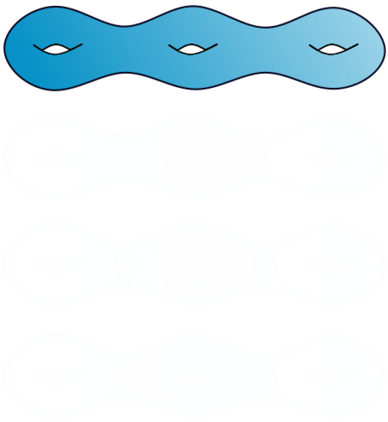


3-Manifolds

Def: Topological spaces which are locally homeomorphic to $\mathbb{R}^3$.



Compression bodies



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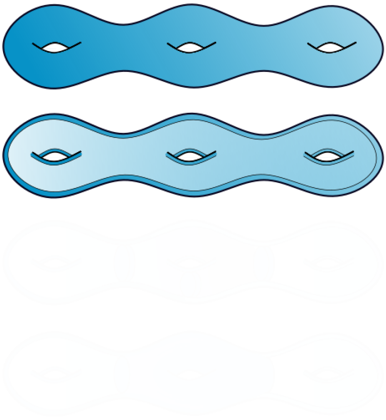

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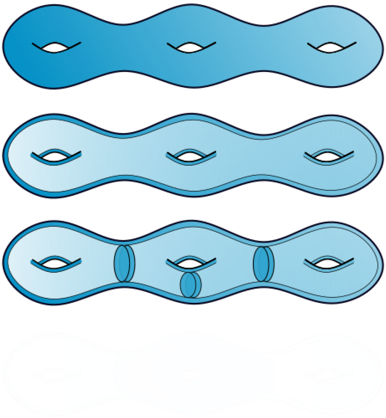

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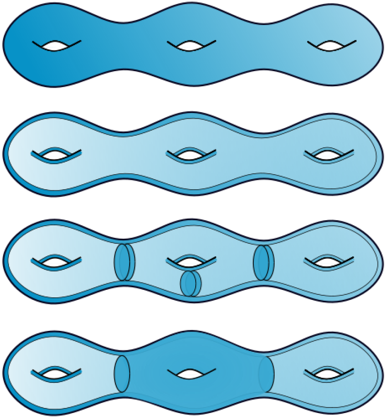
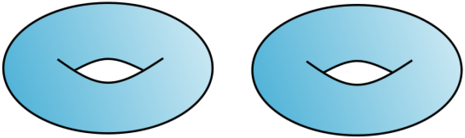


Call the "outside" surface $\partial_+ C$ and let $\partial_- C := \partial C \setminus \partial_+ C$.

Dual construction of compression bodies



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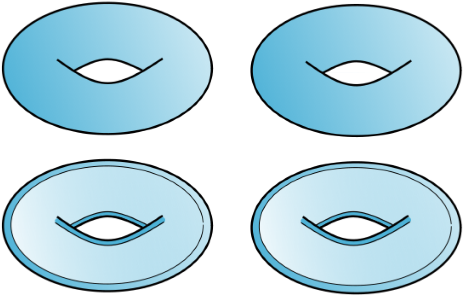

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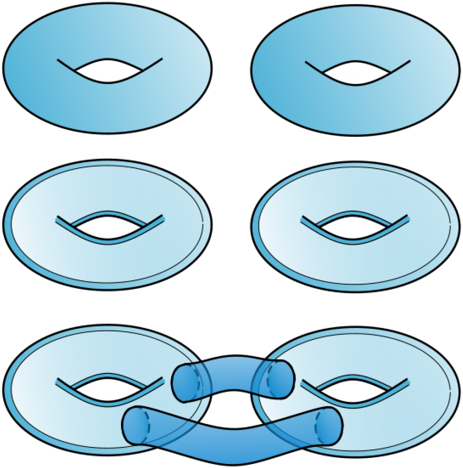

Here we start with $\partial_- C$ and build up to $\partial_+ C$ rather than the other way around.

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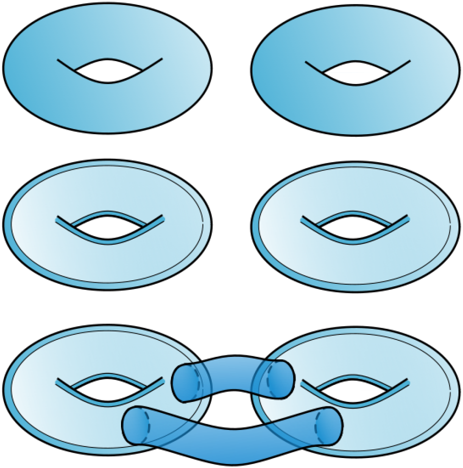

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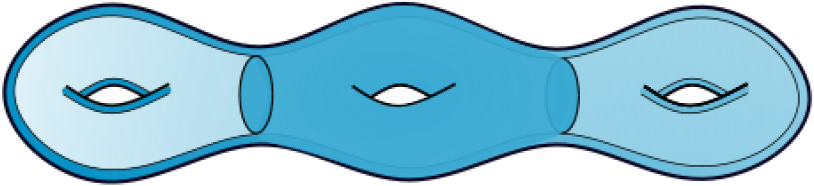
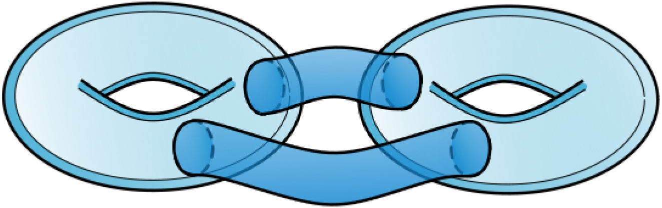
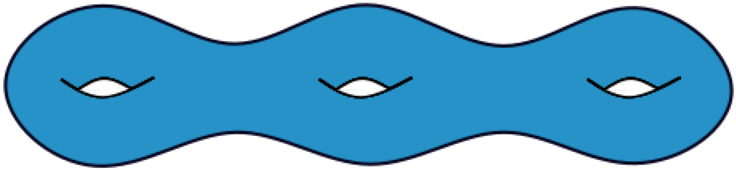


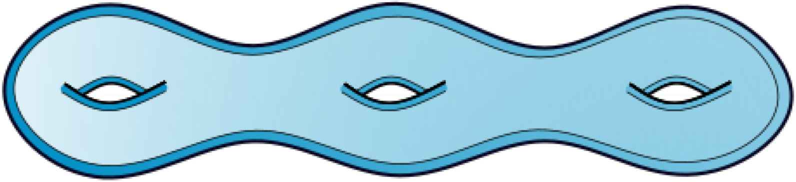


Types of compression bodies

Def: A **handlebody** is a compression body C where $\partial_- C = \emptyset$.

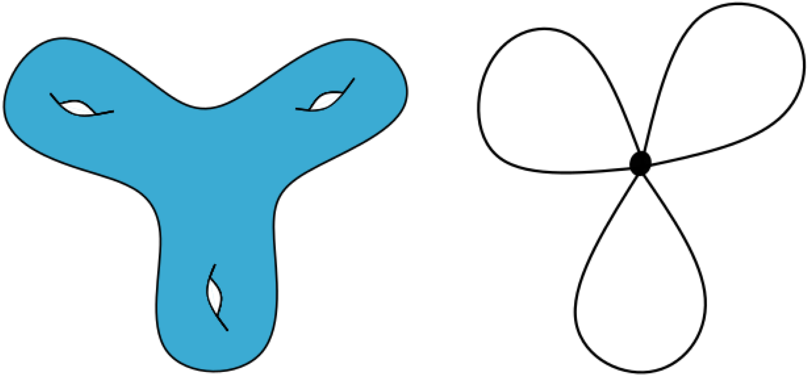


Def: A **trivial compression body** is $S \times I$ for some surface S.



Nice Fact

Remark: All handlebodies retract to a graph known as the **spine** of the handlebody



Heegaard splitting

Def: A **Heegaard splitting** of a 3-manifold M is a pair of compression bodies V and W such that

- $V \cap W = \partial_+ V = \partial_+ W = F$
- $V \cup W = M$.

This is denoted (F, V, W),(M, F), or $V \cup_F W$.

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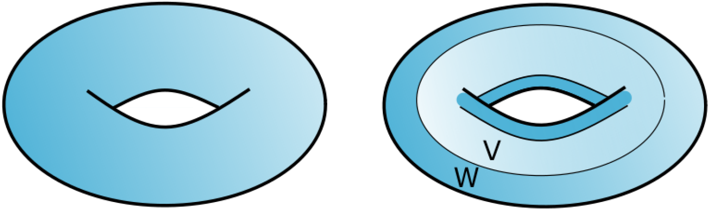
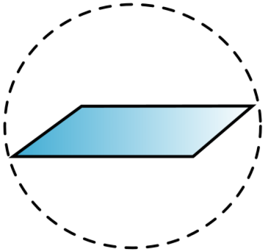


Heegaard splitting

We can split S^3 into two 3-balls (handlebodies of genus 0).



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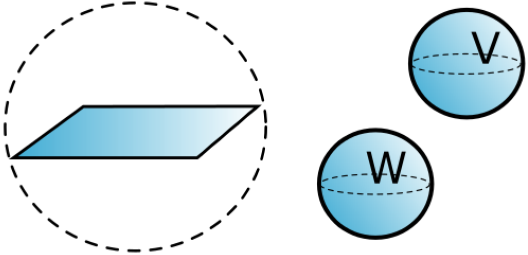
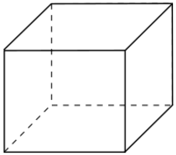


Heegaard splitting

We can split T^3 ($S^1 \times S^1 \times S^1$) into two handlebodies of genus 3.



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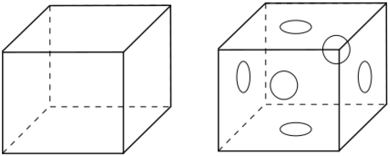

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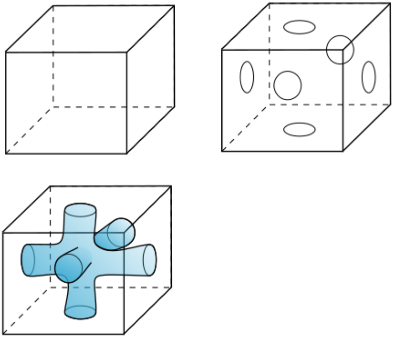

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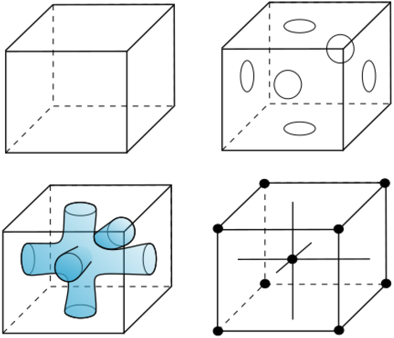

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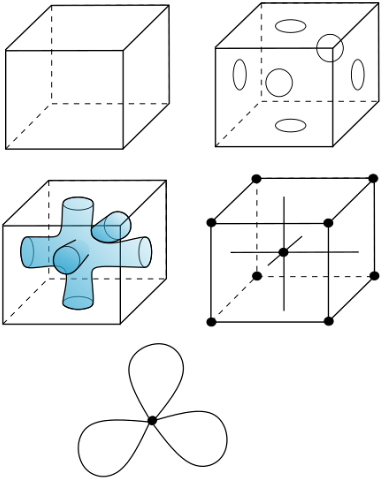

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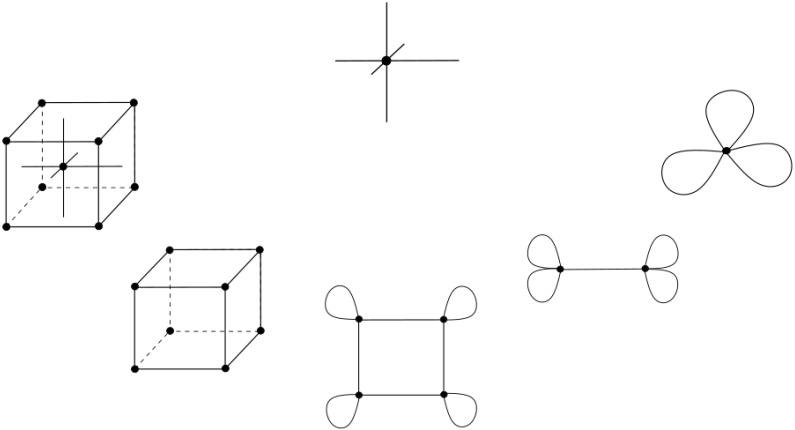

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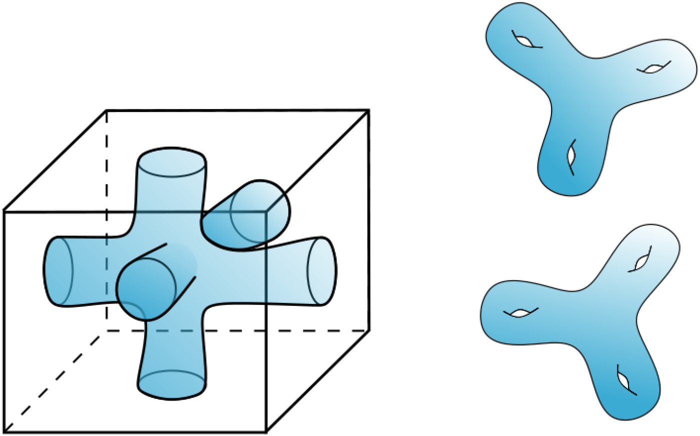
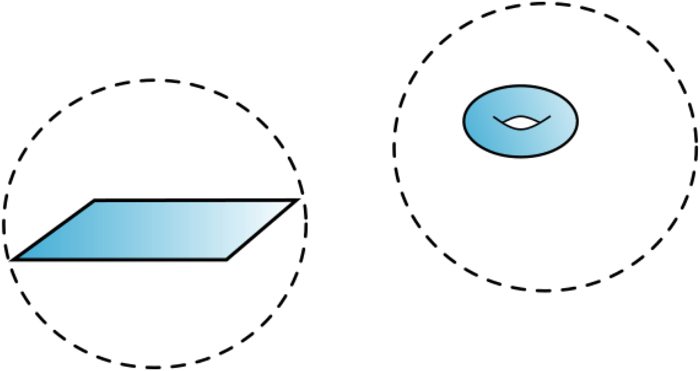


Elementary stabilization of a Heegaard splitting

Def: An **elementary stabilization** of a Heegaard splitting (M, F) is a connect sum between the pairs (M, F) and (S^3, T) where T is an unknotted torus.



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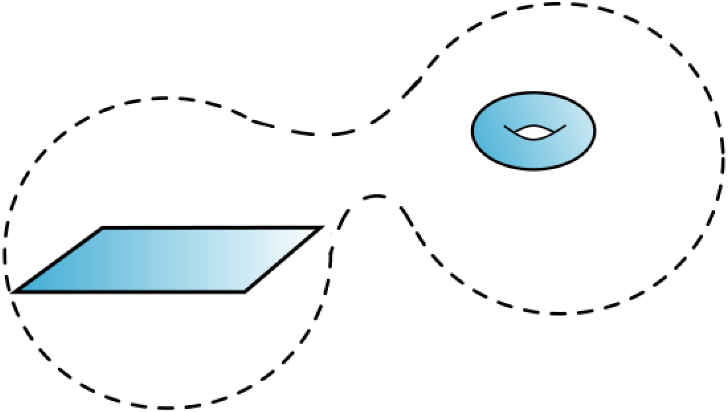

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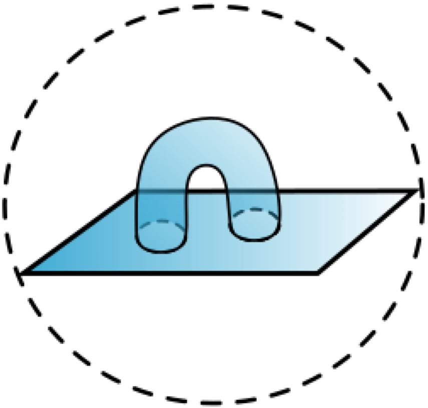
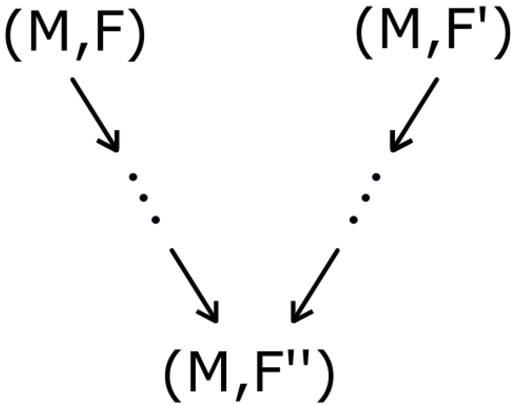


Stably equivalent

Def: Two Heegaard splittings (M, F) and (M, F') of the same 3-manifold M are **stably equivalent** if there exists some Heegaard splitting (M, F'') such that:



Stabilization Theorem

Theorem (Reidemeister, Singer)

Any two Heegaard splittings of an orientable, closed 3-manifold are stably equivalent.

Proof Idea/Beginnings: Let (F, V, W) and (F', V', W') be Heegaard splittings of an orientable, closed 3-manifold M. Isotopy V' and W to be disjoint.

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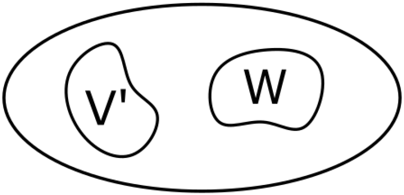

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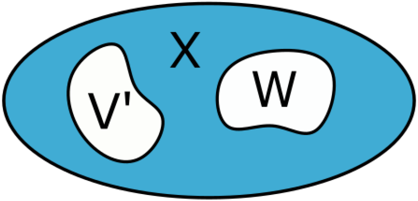

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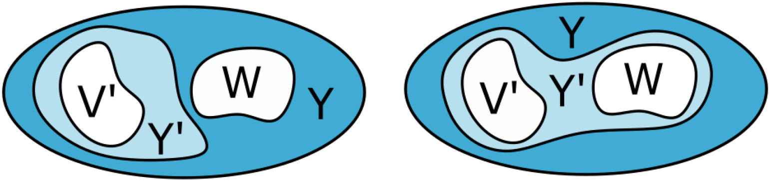
Let (S, Y, Y') be a Heegaard splitting of X.



Reference

Lei Fengchun
On Stability of Heegaard Splittings
(1999)